Geometric inequalities outside a convex set in a Riemannian manifold

By

Keomkyo Seo

Abstract

Let $M$ be an $n$-dimensional complete simply connected Riemannian manifold with nonpositive sectional curvature for $n = 2, 3$ and 4. We prove the following Faber-Krahn type inequality for the first eigenvalue $\lambda_1$ of the mixed boundary problem. A domain $\Omega$ outside a closed convex subset $C$ in $M$ satisfies

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

with equality if and only if $\Omega$ is isometric to the half ball $\Omega^*$ in $\mathbb{R}^n$, whose volume is equal to that of $\Omega$. We also prove the Sobolev type inequality outside a closed convex set $C$ in $M$.

1. Introduction

One of the most important inequalities in geometric analysis is the Faber-Krahn inequality. In the 1920’s, for a bounded domain $\Omega \subset \mathbb{R}^n$, Faber and Krahn proved independently the following inequality

$$(1.1) \quad \lambda_1(\Omega) \geq \lambda_1(\Omega^*),$$

where equality holds if and only if $\Omega$ is a ball (See [1]). Here $\lambda_1$ denotes the first Dirichlet eigenvalue and $\Omega^*$ is a ball of the same $n$-dimensional volume as $\Omega$. For the first Neumann eigenvalue $\mu_1$, in 1954 Szegö[10] showed that for a simply connected domain $\Omega \subset \mathbb{R}^2$

$$\mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where $\Omega^*$ is as above and equality holds if and only if $\Omega$ is a disk. It should be mentioned that $\mu_1$ is the first positive eigenvalue of the Neumann boundary problem. Two years later Weinberger [11] generalized the inequality for $\Omega \subset \mathbb{R}^n$, $n \geq 2$. On the other hand, for the first eigenvalue $\lambda_1$ of the mixed boundary problem, Nehari [8, Theorem III] proved (1.1) for a simply connected bounded domain $\Omega \subset \mathbb{R}^2$ satisfying that a subarc $\alpha \subset \partial \Omega$ is concave with respect to
In this case $\Omega^*$ is a half disk of the same area as $\Omega$. Equality holds if and only if $\Omega$ is a half disk. In Section 2, we prove the Faber-Krahn type inequality (Theorem 2.1) extending Nehari’s result to a Riemannian manifold case.

In [9], the author has proved the Sobolev type inequality outside a closed convex set in a nonpositively curved surface. In Section 3, we study Sobolev type inequality outside a closed convex set in a 3 and 4-dimensional Riemannian manifold with nonpositive sectional curvature.

The key ingredient in the proofs of our theorems is the following relative isoperimetric inequality.

**Theorem 1** ([2], [3], [5], [9]). Let $M$ be an $n$-dimensional complete simply connected Riemannian manifold with nonpositive sectional curvature for $n=2, 3$ and $4$, and let $C \subset M$ be a closed convex set with smooth boundary. Then for a domain $\Omega \subset M \sim C$ we have

\begin{equation}
\frac{1}{2} n^n \omega_n \text{Vol}(\Omega)^{n-1} \leq \text{Vol}(\partial \Omega \sim \partial C)^n,
\end{equation}

where equality holds if and only if $\Omega$ is a Euclidean half ball.

Recently Choe-Ghomi-Ritoré [4] have proved that this inequality holds for a domain in $\mathbb{R}^n$.

**Theorem 2** ([4]). Let $C \subset \mathbb{R}^n$ be a closed convex set with smooth boundary. Then for a domain $\Omega \subset \mathbb{R}^n \sim C$, (1.2) is still true and equality holds if and only if $\Omega$ is a Euclidean half ball.

## 2. Faber-Krahn type inequality

Let $\Omega$ be a bounded domain outside a closed convex subset $C$ with smooth boundary in an $n$-dimensional Riemannian manifold $M$. The Laplacian operator $\Delta$ acting on functions is locally given by

\[
\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right),
\]

where $(x^1, \ldots, x^n)$ is a local coordinate system, $(g^{ij})$ is the inverse of the metric tensor $(g_{ij})$, and $g = \det(g_{ij})$. We consider the mixed eigenvalue problem as follows:

\[
\Delta u + \lambda u = 0 \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega \sim \partial C \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \cap \partial C,
\]

where $\nu$ is the outward unit normal to $\partial \Omega$ along $\partial \Omega \cap \partial C$ and $\sim$ denotes the set exclusion operator. Then, using the divergence theorem, we see that the first eigenvalue $\lambda_1(\Omega)$ of the mixed boundary problem satisfies

\[
\lambda_1(\Omega) = \inf_{u \in H^1_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}.
\]
where $H^1_0(\Omega)$ is the Sobolev space such that $u \in H^1_0(\Omega)$ vanishes on $\partial \Omega \sim \partial C$. We note that $u \in H^1_0(\Omega)$ need not vanish on $\partial \Omega \cap \partial C$.

First we show that the first eigenvalue of the mixed boundary problem for a half ball in space form $M^n(\kappa)$ is equal to that of Dirichlet boundary problem for a ball in $M^n(\kappa)$, where $M^n(\kappa)$ denotes an $n$-dimensional complete Riemannian manifold of constant sectional curvature $\kappa$.

**Proposition 2.1.** Let $\lambda_1(B_+(r))$ be the first mixed eigenvalue of a half ball $B_+(r)$ with radius $r$ in $M^n(\kappa)$ and $\lambda_1(B(r))$ the first eigenvalue of the Dirichlet boundary problem of a ball $B(r)$ with the same radius $r$ in $M^n(\kappa)$. If $\kappa > 0$ assume $r < 1/\sqrt{\kappa}$. Then we have

$$\lambda_1(B_+(r)) = \lambda_1(B(r))$$

**Proof.** First let $\phi$ be an eigenfunction of $B_+(r)$ associated with $\lambda_1(B_+(r))$. Then,

$$\Delta \phi + \lambda_1(B_+(r)) = 0 \text{ in } B_+(r)$$

$$\phi = 0 \text{ on } \partial B_+(r) \sim \partial \mathbb{H}$$

$$\frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial \mathbb{H},$$

where $\partial \mathbb{H}$ denotes the boundary of the half space, which has flat geodesic curvature. We can extend the eigenfunction $\phi$ to $\tilde{\phi}$ defined on $B(r)$ by reflecting $\phi$ across $\partial \mathbb{H}$.

Using $\lambda_1(B(r)) = \inf_{u \in H^1_0(B(r))} \frac{\int_{B(r)} |\nabla u|^2}{\int_{B(r)} u^2}$, we have

$$\lambda_1(B(r)) \leq \frac{\int_{B(r)} |\nabla \tilde{\phi}|^2}{\int_{B(r)} \tilde{\phi}^2} = \frac{\int_{B_+(r)} |\nabla \phi|^2}{\int_{B_+(r)} \phi^2} = \lambda_1(B_+(r)),$$

where $H^1_0(B(r))$ is the Sobolev space on $B(r)$. Conversely let $\psi$ be an eigenfunction of the Dirichlet problem in a ball $B$ associated with $\lambda_1(B(r))$, that is,

$$\Delta \psi + \lambda_1(B(r)) = 0 \text{ in } B(r)$$

$$\psi = 0 \text{ on } \partial B(r).$$

Since $\psi$ is a radial function, $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial \mathbb{H}$. Hence $\psi$ satisfies the boundary condition for the mixed eigenvalue problem. We immediately get

$$\lambda_1(B_+(r)) \leq \frac{\int_{B_+(r)} |\nabla \psi|^2}{\int_{B_+(r)} \psi^2} = \frac{\int_{B(r)} |\nabla \psi|^2}{\int_{B(r)} \psi^2} = \lambda_1(B(r)).$$

Therefore we have $\lambda_1(B_+(r)) = \lambda_1(B(r))$ by (2.1) and (2.2).

We need the following well-known lemma before we prove our theorems.
Lemma 2.1. Let $\Omega$ be a domain in an $n$-dimensional Riemannian manifold $M$ and let $f$ be any eigenfunction with the first eigenvalue $\lambda_1$ for mixed eigenvalue problem. Then $f$ is strictly positive or strictly negative in $\Omega$.

Proof. Note that
\[ \lambda_1(\Omega) = \frac{\int_{\Omega} |\nabla f|^2}{\int_{\Omega} f^2} = \frac{\int_{\Omega} |\nabla |f||^2}{\int_{\Omega} f^2}. \]

It follows that $|f|$ also is an eigenfunction associated with $\lambda_1$ and $|f| \in C^2(\Omega) \cap C^0(\Omega)$ by elliptic regularity theory [7]. We also have $\Delta |f| = -\lambda_1 |f| \leq 0$. Using maximum principle we have $|f| > 0$ in $\Omega$ and hence $f > 0$ or $f < 0$ in $\Omega$.

We now prove the following Faber-Krahn type inequality for the mixed eigenvalue problem using symmetrization and relative isoperimetric inequality.

Theorem 2.1. Let $M$ be an $n$-dimensional complete simply connected Riemannian manifold with nonpositive sectional curvature for $n=2, 3$ and $4$, and let $C \subset M$ be a closed convex set with smooth boundary. Then for a domain $\Omega \subset M \sim C$, we have
\[ \lambda_1(\Omega) \geq \lambda_1(\Omega^*), \]
where $\Omega^*$ is a half ball in $\mathbb{R}^n$, whose volume is equal to that of the domain $\Omega$. Equality holds if and only if the domain $\Omega$ is isometric to the half ball $\Omega^*$ in $\mathbb{R}^n$.

Proof. Let $f$ be the first eigenfunction of $\Omega$, that is,
\[ \Delta f + \lambda_1(\Omega)f = 0 \text{ in } \Omega \]
\[ f = 0 \text{ on } \partial \Omega \sim \partial C \]
\[ \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial \Omega \cap \partial C. \]

We may assume that $f$ is nonnegative by lemma 2.1. Consider the set $\Omega_t = \{ x \in \Omega : f(x) > t \}$ and $\Gamma_t = \{ x \in \Omega : f(x) = t \}$. Using a symmetrization procedure, we construct the concentric geodesic half ball $\Omega_t^*$ in $\mathbb{R}^n$ such that $\text{Vol}(\Omega_t^*) = \text{Vol}(\Omega_t)$ for each $t$, and $\Omega_0^* = \Omega^*$. We define a function $F : \Omega^* \to \mathbb{R}_+$ such that $F$ is a radially decreasing function and $\partial \Omega_t^* \sim \partial H = \{ x \in \Omega^* : F(x) = t \}$.

Then it suffices to prove
\[ \int_{\Omega} f^2 dv = \int_{\Omega^*} F^2 dv, \]
\[ \int_{\Omega} |\nabla f|^2 dv \geq \int_{\Omega^*} |\nabla F|^2 dv. \]
For (2.4), using the co-area formula [6],
\[
\int_{\Omega} f^2 \, dv = \int_0^\infty \int_{\Gamma_t} \frac{t^2}{|\nabla f|} \, dA_t \, dt = \int_0^\infty t^2 \left( \int_{\Gamma_t} \frac{dA_t}{|\nabla f|} \right) \, dt \\
= - \int_0^\infty t^2 \frac{d}{dt} \text{Vol}(\Omega_t) \, dt = - \int_0^\infty t^2 \frac{d}{dt} \text{Vol}(\Omega^*_t) \, dt = \int_{\Omega^*} F^2 \, dv,
\]
where \(dA_t\) is the \((n-1)\)-dimensional volume element on \(\Gamma_t\). Here we have used the identity
\[
\frac{d}{dt} \text{Vol}(\Omega_t) = - \int_{\Gamma_t} |\nabla f|^{-1} \, dA_t.
\]
For (2.5), using Hölder inequality we have
\[
\int_{\Gamma_t} dA_t = \int_{\Gamma_t} |\nabla f|^{1/2} |\nabla |^{-1/2} \, dA_t \\
\leq \left( \int_{\Gamma_t} |\nabla f| \right)^{1/2} \left( \int_{\Gamma_t} |\nabla f|^{-1} \right)^{1/2} \\
= \left( \int_{\Gamma_t} |\nabla f| \right)^{1/2} \left( - \frac{d}{dt} \text{Vol}(\Omega_t) \right)^{1/2}.
\]
From the relative isoperimetric inequality (1.2) as mentioned in the introduction, we see that
\[
(2.6) \quad \int_{\Gamma_t} |\nabla f| \, dA_t \geq \frac{\text{Vol}(\Omega_t)^2}{\int_{\Gamma_t} |\nabla f|^{-1} \, dA_t} = \int_{\Gamma^*_t} |\nabla F| \, dA^*_t,
\]
where \(\Gamma^*_t = \{ x \in \Omega^*: F(x) = t \}\), and \(dA^*_t\) is the \((n-1)\)-dimensional volume element on \(\Gamma^*_t\). Integrating in \(t\), we get (2.5). To have equality, the second inequality in (2.6) should become equality. Since equality in the relative isoperimetric inequality holds if and only if \(\Omega\) is isometric to a half ball in \(\mathbb{R}^n\), we get the conclusion.

Using [4], we can also prove the following.

**Theorem 2.2.** Let \(C \subset \mathbb{R}^n\) be a closed convex set with smooth boundary. Then for a domain \(\Omega \subset \mathbb{R}^n \sim C\), we have
\[
(2.7) \quad \lambda_1(\Omega) \geq \lambda_1(\Omega^*),
\]
where \(\Omega^*\) is a half ball in \(\mathbb{R}^n\), whose volume is equal to that of the domain \(\Omega\). Equality holds if and only if the domain \(\Omega\) is isometric to the half ball \(\Omega^*\) in \(\mathbb{R}^n\).
3. Sobolev type inequality

In this section we prove Sobolev type inequality outside a closed convex set in a Riemannian manifold.

**Theorem 3.1.** Let $M$ be an $n$-dimensional complete simply connected Riemannian manifold with nonpositive sectional curvature for $n=2, 3$ and $4$. Let $C \subset M$ be a closed convex set. Then we have

$$\frac{1}{2} n^n w_n \left( \int_{M \sim C} |f|^{\frac{n}{n-1}} dA \right)^{n-1} \leq \left( \int_{M \sim C} |\nabla f| dA \right)^n, f \in W^{1,1}_0 (M \sim C).$$

Equality holds if and only if up to a set of measure zero, $f = c \chi_D$ where $c$ is a constant and $D$ is a half ball in $\mathbb{R}^n$.

**Proof.** For simplicity, we assume $f \geq 0$. By the co-area formula

$$\int_M |\nabla f| dv = \int_0^\infty \text{Area}(f = \sigma) d\sigma.$$

We apply the relative isoperimetric inequality (1.2) to obtain

$$\int_M |\nabla f| dv = \int_0^\infty \text{Area}(f = \sigma) d\sigma \geq n \left( \frac{\omega_n}{2} \right)^\frac{1}{n-1} \int_0^\infty \text{Vol}(f > \sigma)^{\frac{n-1}{n}} d\sigma.$$

Since we have

$$\int_M |f|^{\frac{n}{n-1}} dv = \int_0^\infty \text{Vol}(f > \rho) d\rho = \frac{n}{n-1} \int_0^\infty \text{Vol}(f > \sigma)^{\frac{1}{n-1}} d\sigma,$$

it suffices to show that

$$\int_0^\infty \text{Vol}(f > \sigma)^{\frac{n-1}{n}} d\sigma \geq \left( \frac{n}{n-1} \right)^{\frac{n-1}{n}} \left( \int_0^\infty \text{Vol}(f > \sigma)^{\frac{1}{n-1}} d\sigma \right)^{\frac{n-1}{n}}.$$

Define

$$F(\sigma) := \text{Vol}(f > \sigma),$$

$$\varphi(t) := \int_0^t F(\sigma)^{\frac{n-1}{n}} d\sigma,$$

$$\psi(t) := \left( \int_0^t F(\sigma)^{\frac{1}{n-1}} d\sigma \right)^{\frac{n-1}{n}}.$$

Then we can see that $\varphi(0) = \psi(0) = 0$. Since $F(\sigma)$ is monotone decreasing, we obtain

$$\varphi'(t) \geq \left( \frac{n}{n-1} \right)^{\frac{n-1}{n}} \psi'(t).$$

It follows that

$$\varphi(\infty) \geq \left( \frac{n}{n-1} \right)^{\frac{n-1}{n}} \psi(\infty).$$
Moreover it is easy to see that equality holds if and only if \( f = c\chi_D \) where \( c \) is a constant and \( D \) is a half ball in \( \mathbb{R}^n \).

Applying the same arguments as in the proof of the above theorem and the relative isoperimetric inequality (1.2), we also have the following theorem.

**Theorem 3.2.** Let \( C \subset \mathbb{R}^n \) be a closed convex set with smooth boundary. Then we have

\[
\frac{1}{2} n^w_n \left( \int_{\mathbb{R}^n \setminus C} |f|^{\frac{n}{n-1}} dA \right)^{\frac{n-1}{n}} \leq \left( \int_{\mathbb{R}^n \setminus C} |\nabla f| dA \right)^n, \quad f \in W^{1,1}_0 (\mathbb{R}^n \sim C).
\]

Equality holds if and only if up to a set of measure zero, \( f = c\chi_D \) where \( c \) is a constant and \( D \) is a half ball in \( \mathbb{R}^n \).

**Remark.** In our Theorem 3.1 and 3.2, the function \( f \) may not vanish on \( \partial C \). It is sufficient that \( f \) is compactly supported in the relative topology on \( S \sim C \) for a closed convex set \( C \subset S \).

References


